Fourier Transforms on $\mathbb{R}$

The Fourier synthesis equation

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{2\pi jft} df$$

and analysis equation

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-2\pi jft} dt$$

say that $x(t)$ and $X(f)$ form a Fourier transform pair. x is the time domain representation of the signal, and X is the frequency domain representation. They are really the same signal, just viewed from different perspectives.

Another interpretation is that $X(f_0)$ specifies in some sense how similar $x(t)$ is to the complex exponetial $e^{2\pi jf_0t}$. The analysis integral is the inner product of x and $e^{2\pi jf_0t}$. Then the synthesis equation says that $x(t)$ is a linear combination of complex exponentials, and $X(f_0)$ specifies how much of $e^{2\pi jf_0t}$ should be included in the recipe for $x(t)$.

In principle, we can always find the Fourier transform (FT) pairs, if they exist, through direct evaluation of the Fourier integral. This is tedious and in many cases nearly impossible. So we use the same technique as we do for finding derivatives. We have a short list of FT pairs, and a number a rules to build new FT pairs from old ones.

In what follows, time domain is on the left, and frequency domain is on the right. Some transformation in one domain will have a corresponding transformation in the other domain.

- Fourier Transform Pair

$$\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} X(f) e^{2\pi jft} df \\
\int_{-\infty}^{\infty} x(t) e^{-2\pi jft} dt &= X(f) \\
\mathcal{F}\{x(t)\} &= X(f) \\
x(t) &= \mathcal{F}^{-1}\{X(f)\} \\
x(t) &\Leftrightarrow X(f)
\end{aligned}$$

- Inversion or Duality ($\mathcal{F}\{x(t)\} = X(f)$ and imagine that X is a function of time instead of frequency).[1]

$$X(t) \Leftrightarrow x(-f)$$

- Linearity ($\alpha, \beta \in \mathbb{C}$ are constant w.r.t. t and f)

$$\alpha x(t) + \beta y(t) \Leftrightarrow \alpha X(f) + \beta Y(f)$$

[1] Use this to remember the sign changes in other rules.

- Conjugation and Reflection
  - Complex Conjugation

$$x^*(t) \Leftrightarrow X^\dagger(f) = X^*(-f)$$

  - Reflection

$$x(-t) \Leftrightarrow X(-f)$$

  - Hermitian Conjugation

$$x^\dagger(t) = x^*(-t) \Leftrightarrow X^*(f)$$

  - Symmetries
    * If $x(t)$ is purely real, then $X(f)$ is hermitian

$$x(t) = x^*(t) \Longleftrightarrow X(f) = X^\dagger(f)$$

    * If $x(t)$ is purely imaginary, then $X(f)$ is antihermitian

$$x(t) = -x^*(t) \Longleftrightarrow X(f) = -X^\dagger(f)$$

    * If $x(t)$ is even, then $X(f)$ is even

$$x(t) = x(-t) \Longleftrightarrow X(f) = X(-f)$$

    * If $x(t)$ is odd, then $X(f)$ is odd

$$x(t) = -x(-t) \Longleftrightarrow X(f) = -X(-f)$$

    * If $x(t)$ is hermitian, then $X(f)$ is purely real

$$x(t) = x^\dagger(t) \Longleftrightarrow X(f) = X^*(f)$$

    * If $x(t)$ is antihermitian, then $X(f)$ is purely imaginary

$$x(t) = -x^\dagger(t) \Longleftrightarrow X(f) = -X^*(f)$$

- Shifting[2]
  - Time Shift ($t_0 \in \mathbb{R}$ is a constant)

$$x(t - t_0) \Leftrightarrow e^{-2\pi j f t_0} X(f)$$

  - Modulation or Frequency Shift ($f_0 \in \mathbb{R}$ is a constant)

$$e^{2\pi j f_0 t} x(t) \Leftrightarrow X(f - f_0)$$

[2]Note the different signs in the exponential.

- Scaling ($a \in \mathbb{R},\ a \neq 0$ is a constant)

$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

- Differentiation[3]
  - Time Differentiation

$$\frac{d}{dt}x(t) \Leftrightarrow 2\pi j f X(f)$$

  - Power Scaling or Frequency Differentiation

$$tx(t) \Leftrightarrow \frac{-1}{2\pi j}\frac{d}{df}X(f)$$

- Convolution
  - Time Convolution

$$(x * h)(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda \Leftrightarrow X(f)H(f)$$

  - Time Multiplication or Frequency Convolution

$$x(t)h(t) \Leftrightarrow (X * H)(f) = \int_{-\infty}^{\infty} X(\lambda)H(f-\lambda)d\lambda$$

  - Time Cross-correlation

$$(x \star h)(t) = (x^{\dagger} * h)(t) = \int_{-\infty}^{\infty} x^*(\lambda)h(t+\lambda)d\lambda \Leftrightarrow X^*(f)H(f)$$

  - Time Autocorrelation

$$(x \star x)(t) = (x^{\dagger} * x)(t) = \int_{-\infty}^{\infty} x^*(\lambda)x(t+\lambda)d\lambda \Leftrightarrow |X(f)|^2$$

- Parseval's Identities or Unitarity[4]
  - Inner Product

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$$

  - Energy or Norm

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

[3] Note the difference in signs.
[4] "Unitary" means "preserves inner products (and norms)."

- Area under the curve
  - DC or Average Value

$$\int_{-\infty}^{\infty} x(t)dt = X(0)$$

  - Value at $t = 0$

$$x(0) = \int_{-\infty}^{\infty} X(f)df$$

- Radian Frequency ($\omega = 2\pi f$)
  - Synthesis

$$x(t) = \int_{-\infty}^{\infty} X_{rad}(\omega)e^{j\omega t}\frac{d\omega}{2\pi}$$

  - Analysis

$$X_{rad}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

  - Equivalence

$$X(f) = X_{rad}\left(2\pi f\right)$$
$$X_{rad}(\omega) = X\left(\frac{\omega}{2\pi}\right)$$